Noetherian rings

An important property in commutative algebra is finite generation of ideals.

For a polynomial ring in finitely many variables over a field $k[x_1, \dots, x_n]$, the fact (which we will show) that every ideal is finitely generated is equivalent to the fact that every variety in $\mathbb{A}^n$ is the intersection of finitely many hypersurfaces (i.e. the zero locus of a single polynomial).

Def: A ring R is <u>Noetherian</u> if every ideal of R is finitely generated.

Equivalently, R is Noetherian if its ideals satisfy the ascending chain condition:

Claim: R is Noetherian iff every strictly increasing chain of ideals terminates.

Pf: If I is not finitely generated, choose $f_1 \in I$, $f_2 \in I \setminus (f_1)$, $f_n \in I \setminus (f_1, \dots, f_{n-1})$. Then we get an infinite chain of ideals

$$(f_1) \subsetneq (f_1, f_2) \subsetneq \dots$$

Conversely, if $I_1 \subsetneq I_2 \subsetneq \dots$ is a chain of ideals and $I = \bigcup I_i$ is finitely generated, then all of the generators are in one of the I_j, so $I = I_j$. $\square$

Ex: Some familiar Noetherian rings: all fields, $\mathbb{Z}$, $\mathbb{Z}[x]$

Thm: (Hilbert Basis Theorem) If R is a Noetherian ring, then $R[x]$ is Noetherian.

(Note that this implies $R[x_1, \dots, x_n]$ is also Noetherian by induction.)

Pf: First, some terminology: If $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \in R[x]$, w/ $a_n \neq 0$, then a_n is the <u>initial coefficient</u>, $a_n x^n$ is the <u>initial term</u>.

Let $I \subseteq R[x]$ be an ideal. Choose a sequence $f_1, f_2, \dots \in I$ as follows: let $f_1 \in I$ be a nonzero elt of least degree,

and let f_{n+1} be an element of least degree in $I \setminus (f_1, \dots, f_n)$

If $(f_1, \dots, f_n) = I$, we're done.

Let a_j be the initial coefficient of f_j. Set $J = (a_1, a_2, \dots) \subseteq R$. Since R Noetherian take m to be the smallest integer s.t. $J = (a_1, \dots, a_m)$.

Claim: $I = (f_1, \dots, f_m)$.

Otherwise consider f_{m+1}. Then $a_{m+1} = \sum_{j=1}^{m} u_j a_j$ for some $u_j \in R$.

Since $\deg f_{m+1} \geq \deg f_j$ for $j \leq m$, we can define

$$g = \sum_{j=1}^{m} u_j f_j x^{\deg f_{m+1} - \deg f_j} \in (f_1, \dots, f_m).$$

$f_{m+1} - g \in I \setminus (f_1, \dots, f_m)$ and has degree strictly less than the degree of f_{m+1}, which is a contradiction. $\square$

Cor: If R is Noetherian, and S a finitely generated R-algebra, then S is Noetherian.

Pf: $S = R[a_1, \dots, a_n]$, so $R[x_1, \dots, x_n]$, which is Noetherian, surjects onto S. Thus, any I in S is generated by the images of generators of its preimage. $\square$

Noetherian Modules

An R-module M is <u>Noetherian</u> if every submodule is finitely generated. (Equivalently, if it has the ascending chain condition on submodules or if every collection of submodules has a maximal element — see HW.)

Prop: If R is a Noetherian ring and M a finitely generated R-module, then M is Noetherian.

Pf: Let $f_1, \ldots, f_n$ be the generators of M and $N \subseteq M$ a submodule.

If $n=1$, consider the map $R \twoheadrightarrow M$ sending $1 \mapsto f_1$. Then the preimage of N is an ideal, so the images of its generators generate N.

If $n>1$, by induction, M/Rf_1 is Noetherian. So if $\bar{N}$ is the image of N in the quotient, $\bar{N}$ is f.g. by the images of $g_1, \ldots, g_s$.

$N \cap Rf_1$ is a submodule of Rf_1, so it's f.g. by $h_1, \ldots, h_r$.

Thus, if $a \in N$, $\bar{a}$ is a lin comb. of the g_i, so a is gen by the g_i and h_j. $\square$